

A new comparison theorem of multidimensional BSDEs

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Abstract In this paper, we first study a property about the generator g of Backward Stochastic Differential Equation (BSDE) when the price of contingent claims can be represented by a multi-dimensional BSDE in the no-arbitrage financial market. Furthermore, motivated by the behavior of agent in finance market, we introduce a new total order \succsim^q on \mathbb{R}^n and obtain a necessary and sufficient condition for comparison theorem of multidimensional BSDEs under this order. We also give some further results for special order \succsim^q .

Keywords Backward stochastic differential equation, Comparison theorem, Viability property

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1 Preliminaries

In this section, we shall introduce some notations and assumptions which are needed in the following analysis.

Let (Ω, \mathcal{F}, P) be a probability space and $(W_t)_{t \geq 0}$ a standard d -dimensional Brownian motion defined on this probability space. Furthermore, let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by the Brownian motion $(W_t)_{t \geq 0}$, that is $\mathcal{F}_t = \sigma(W_s; s \leq t)$. We define the usual P -augmentation to each \mathcal{F}_t such that $(\mathcal{F}_t)_{t \geq 0}$ is right continuous and complete. We denote by \mathbb{R}^n the n -dimensional Euclidean space, equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. We also denote by $\mathbb{R}^{n \times d}$ the collection of all $n \times d$ real matrices, and for matrix $z = (z_{ij})_{n \times d}$, we denote $z_i := (z_{i1}, \dots, z_{id})^T$ and $|z| := \sqrt{\text{tr}(zz^T)}$, where z^T represents the transpose of z . Now, we define the following usual spaces of random variables or processes:

- $L^2(\Omega, \mathcal{F}_t, P; K) = \{\xi \mid \xi \text{ is } K\text{-valued } \mathcal{F}_t\text{-measurable random variable and } \mathbb{E}[\xi^2] < \infty\}$, where K is a subset of \mathbb{R}^n ;
- $\mathcal{S}_T^2 = \{\psi \mid \psi \text{ is } \mathbb{R}^n\text{-valued progressively measurable process and } \mathbb{E}[\sup_{0 \leq t \leq T} |\psi_t|^2] < \infty\}$;
- $\mathcal{H}_T^2 = \{\psi \mid \psi \text{ is } \mathbb{R}^{n \times d}\text{-valued progressively measurable process and } \mathbb{E}[\int_0^T |\psi_t|^2 dt] < \infty\}$.

Consider the following Backward Stochastic Differential Equation (BSDE for short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (1)$$

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where ξ is a given n -dimensional random variable, and g is called the generator of BSDE (1), which is defined as $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$, such that the process $(g(t, y, z))_{t \in [0, T]}$ is progressively measurable for each (y, z) in $\mathbb{R}^n \times \mathbb{R}^{n \times d}$. We make the following assumptions (A1)-(A4) throughout this paper:

- (A1) For any $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$, $t \rightarrow g(t, y, z)$ is continuous, P -a.s.;
- (A2) There exists a constant $\mu > 0$, such that for any $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$, we have

$$|g(t, y, z) - g(t, y', z')| \leq \mu(|y - y'| + |z - z'|), \quad P\text{-a.s.};$$

- (A3) $g(t, 0, 0) \in \mathcal{S}_T^2$;
- (A4) $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$.

Under the assumptions (A1)-(A4), Pardoux and Peng [9] showed that BSDE (1) has a unique adapted solution (Y, Z) belonging to $\mathcal{S}_T^2 \times \mathcal{H}_T^2$. The viability property of a stochastic process as a classical notion in stochastic context was first discussed in [1]. Buckdahn et al. [2] studied the viability property of BSDE (1). In Theorem 2.5 of [2], they obtained the Backward Stochastic Viability Property (BSVP for short) for BSDE. In next section, using the BSVP, we study the property about the generator g of BSDE when the price vector of contingent claims can be represented by a multidimensional BSDE in the no-arbitrage financial market. Some other applications of BSDE in financial mathematics can be found for example in [3, 5, 6, 8].

The comparison theorem and related converse comparison theorem for one dimensional BSDEs were important results in the theory of BSDE first due to Peng [10] and Coquet et al. [4] respectively, and later generalized by several authors (for example see [5]). Combining comparison theorem and converse comparison theorem for one-multidimensional BSDEs, we can get the following result: For any $0 \leq u \leq T$, consider the following two BSDEs,

$$Y_t^i = \xi^i + \int_t^u g^i(s, Y_s^i, Z_s^i) ds - \int_t^u Z_s^i dW_s, \quad 0 \leq t \leq u, \quad i = 1, 2, \quad (2)$$

where g^1 and g^2 satisfy (A1)-(A3), then the statements (i') and (ii') are equivalent:

- (i') For any $u \in [0, T]$, $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_u, P; \mathbb{R})$, if $\xi^1 \geq \xi^2$, then $Y_t^1 \geq Y_t^2$, P -a.s., for $t \in [0, u]$.
- (ii') For any $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g^1(t, y, z) \geq g^2(t, y, z)$, P -a.s..

A natural question is whether the above equivalence still holds for multidimensional BSDEs. To answer this question, the key point is how to define "order" or "preference" on \mathbb{R}^n . Hu and Peng [7] considered the case where the "order" $y^1 \geq y^2$ on \mathbb{R}^n is in the sense of $y_i^1 \geq y_i^2$, for all $i = 1, 2, \dots, n$, where y_i^1 and y_i^2 are the i -th component of y^1 and y^2 respectively. They obtained a necessary and sufficient condition of (i') for multidimensional BSDEs. However, in finance, such a preference is not enough to describe the behavior of agents. For example, in financial market, let $y^1 = (y_1^1, \dots, y_n^1)$, $y^2 = (y_1^2, \dots, y_n^2)$ be two portfolios consisting of n -basic contingent claims, q be the price vector of those contingent claims. Agents often like to compare the value $\langle y^1, q \rangle$ and $\langle y^2, q \rangle$ of portfolios. In this case, it is natural to define a total order \succsim^q on \mathbb{R}^n via q . What is comparison theorem under this order? In Section 3, we re-state (i') for multidimensional BSDEs, and obtain a necessary and sufficient condition for comparison theorem of multidimensional BSDEs under the new total order. The result is another application of BSVP. We also give some further results for special total order \succsim^q .

2 BSVP and its Application

Let us recall the definition of BSVP from [2].

Definition 2.1 *Let K be a nonempty, convex closed set in \mathbb{R}^n . Then we call the BSDE (1) enjoys the BSVP in K if: for any $u \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}_u, P; K)$, the unique solution $(Y, Z) \in \mathcal{S}_u^2 \times \mathcal{H}_u^2$ to the BSDE (1) over time interval $[0, u]$, given by*

$$Y_t = \xi + \int_t^u g(s, Y_s, Z_s) ds - \int_t^u Z_s dW_s, \quad 0 \leq t \leq u, \quad (3)$$

satisfies for any $t \in [0, u]$, $Y_t \in K$, P -a.s..

For completeness, we recall the necessary and sufficient condition of BSVP for the BSDEs in [2].

Proposition 2.2 *Let K be a nonempty, convex closed set in \mathbb{R}^n . Suppose that g satisfies (A1)-(A3). Then BSDE (1) enjoys the BSVP in K if and only if for any $(t, z) \in [0, T] \times \mathbb{R}^{n \times d}$ and any $y \in \mathbb{R}^n$ such that $d_K^2(\cdot)$ is twice differentiable at y ,*

$$4 \langle y - \Pi_K(y), g(t, \Pi_K(y), z) \rangle \leq \langle D^2 d_K^2(y) z, z \rangle + C d_K^2(y), \quad P\text{-a.s.}, \quad (4)$$

where $C > 0$ is a constant which does not depend on (t, y, z) , $\Pi_K(y)$ is the projection of y onto K , $d_K(y)$ represents the distance between y and K .

Now we give an application of BSVP in the no-arbitrage financial market. We get a property of g when the price vector of contingent claims can be represented by a multidimensional BSDE (see [5] for details). Suppose that there are n kinds of contingent claims in the market. The price vector of this n contingent claims is a random process $(Y_t)_{0 \leq t \leq T}$ with $Y_t \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Let X^1, X^2 be two different kinds of risk positions. Furthermore, without loss of generality, assume that the terminal value of X^1 is bigger than that of X^2 , in other words, $\langle Y_T, q \rangle \geq 0$, where $q := (X^1 - X^2)/|X^1 - X^2|$. Then we have the following theorem.

Theorem 2.3 *If*

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (5)$$

then

$$-4 \langle y, q \rangle^- \langle q, g(t, y + \langle y, q \rangle^- q, z) \rangle \leq 2 I_{\langle y, q \rangle < 0} \sum_{i,j=1}^n q_i q_j z_i^T z_j + C (\langle y, q \rangle^-)^2, \quad P\text{-a.s.},$$

where $C > 0$ is a constant which does not depend on (t, y, z) .

Proof. Since the market is no-arbitrage, we have if $\langle Y_T, q \rangle \geq 0$, then $\langle Y_t, q \rangle \geq 0$, P -a.s., for any $t \in [0, T]$ (for details see [5]). That is, BSDE (5) enjoys the BSVP in K , where $K := \{y \in \mathbb{R}^n \mid \langle y, q \rangle \geq 0\}$. Clearly, K is a nonempty, convex closed set of \mathbb{R}^n .

If $y \in K$, then $\Pi_K(y) = y, d_K(y) = 0$. If $y \notin K$, we have

$$\begin{cases} \langle \Pi_K(y), q \rangle = 0 \\ \Pi_K(y) - d_K(y)q = y \end{cases}.$$

Let $\Pi_K(y) = (u_1, u_2, \dots, u_n)$, solving the above two equations, we get

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ d_K(y) \end{pmatrix} = \begin{pmatrix} 1 - q_1^2 & -q_1 q_2 & \cdots & -q_1 q_n & q_1 \\ -q_1 q_2 & 1 - q_2^2 & \cdots & -q_2 q_n & q_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -q_1 q_n & -q_2 q_n & \cdots & 1 - q_n^2 & q_n \\ -q_1 & -q_2 & \cdots & -q_n & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ 0 \end{pmatrix},$$

that is, $\Pi_K(y) = y - \langle y, q \rangle q$, $d_K(y) = -\langle y, q \rangle$. Consequently, for any $y \in \mathbb{R}^n$, we get $\Pi_K(y) = y + \langle y, q \rangle^- q$, $d_K(y) = \langle y, q \rangle^-$. Therefore, for $y \in \mathbb{R}^n$, $D^2 d_K^2(y) = 2qq^T I_{\langle y, q \rangle < 0}$. Due to Proposition 2.2, g must satisfies

$$-4\langle y, q \rangle^- \langle q, g(t, y + \langle y, q \rangle^- q, z) \rangle \leq 2I_{\langle y, q \rangle < 0} \sum_{i,j=1}^n q_i q_j z_i^T z_j + C(\langle y, q \rangle^-)^2, \quad P\text{-a.s..}$$

The proof of Theorem 2.3 is completed. \square

3 Comparison Theorem for Multidimensional BSDEs under the Total Order \succsim^q

In this section, we first introduce the definition of a total order on \mathbb{R}^n denoted by \succsim^q and then prove the comparison theorem for multidimensional BSDEs under this total order.

Definition 3.1 Let $q \in \mathbb{R}^n$ be any fixed nonvanishing vector. For any $y^1, y^2 \in \mathbb{R}^n$, we call y^1 bigger (or better) than y^2 under q , denote $y^1 \succsim^q y^2$, if $\langle y^1, q \rangle \geq \langle y^2, q \rangle$.

Remark 3.2 (1) Obviously, $y^1 \succsim^q y^2$ if and only if $y^1 \succsim^{q/|q|} y^2$. So without loss of generality, we assume q be a unit vector in the sequel.

(2) \succsim^q is a total order on \mathbb{R}^n , which can be used to compare any two elements in \mathbb{R}^n .

We now begin to prove the comparison theorem of multidimensional BSDEs under the total order \succsim^q .

For any $0 \leq u \leq T$, consider the following two BSDEs,

$$Y_t^i = \xi^i + \int_t^u g^i(s, Y_s^i, Z_s^i) ds - \int_t^u Z_s^i dW_s, \quad 0 \leq t \leq u, \quad i = 1, 2. \quad (6)$$

Theorem 3.3 Suppose that g^1 and g^2 satisfy (A1)-(A3). Then, the following two statements (i) and (ii) are equivalent.

• (i) For any $u \in [0, T]$, $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_u, P; \mathbb{R}^n)$ such that $\xi^1 \succsim^q \xi^2$, then the unique solutions (Y^1, Z^1) and (Y^2, Z^2) in $\mathcal{S}_u^2 \times \mathcal{H}_u^2$ to BSDEs (6) over time interval $[0, u]$ satisfy

$$Y_t^1 \succsim^q Y_t^2, \quad P\text{-a.s.}, \quad \forall t \in [0, u].$$

• (ii) For any $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$, we have

$$\begin{aligned} & -4\langle y, q \rangle^- \langle q, g^1(t, y + \langle y, q \rangle^- q + y', z) - g^2(t, y', z') \rangle \\ & \leq 2I_{\langle y, q \rangle < 0} \sum_{i,j=1}^n q_i q_j (z_i - z'_i)^T (z_j - z'_j) + C(\langle y, q \rangle^-)^2, \quad P\text{-a.s.}, \end{aligned} \quad (7)$$

where $C > 0$ is a constant independent of (t, y, z) .

Proof. It is obviously that (6) \Leftrightarrow

$$\begin{cases} Y_t^1 - Y_t^2 = \xi^1 - \xi^2 + \int_t^u [g^1(s, Y_s^1, Z_s^1) - g^2(s, Y_s^2, Z_s^2)] ds - \int_t^u (Z_s^1 - Z_s^2) dW_s, & 0 \leq t \leq u, \\ Y_t^2 = \xi^2 + \int_t^u g^2(s, Y_s^2, Z_s^2) ds - \int_t^u Z_s^2 dW_s, & 0 \leq t \leq u. \end{cases}$$

Let

$$\bar{Y}_t = \begin{pmatrix} \bar{Y}_t^1 \\ \bar{Y}_t^2 \end{pmatrix} := \begin{pmatrix} Y_t^1 - Y_t^2 \\ Y_t^2 \end{pmatrix},$$

$$\bar{Z}_t = \begin{pmatrix} \bar{Z}_t^1 \\ \bar{Z}_t^2 \end{pmatrix} := \begin{pmatrix} Z_t^1 - Z_t^2 \\ Z_t^2 \end{pmatrix}, \quad \bar{\xi} = \begin{pmatrix} \bar{\xi}^1 \\ \bar{\xi}^2 \end{pmatrix} := \begin{pmatrix} \xi^1 - \xi^2 \\ \xi^2 \end{pmatrix}.$$

Then (i) is equivalent to the following statement (iii):

- (iii) For any $u \in [0, T]$, $\bar{\xi} = \begin{pmatrix} \bar{\xi}^1 \\ \bar{\xi}^2 \end{pmatrix} \in L^2(\Omega, \mathcal{F}_u, P; \mathbb{R}^{2n})$ such that $\bar{\xi}^1 \succsim^q 0$, the unique solution (\bar{Y}, \bar{Z}) to the following $2n$ -dimensional BSDE (8) over time interval $[0, u]$ satisfies $\bar{Y}_t^1 \succsim^q 0$, P -a.s.,

$$\bar{Y}_t = \bar{\xi} + \int_t^u \bar{g}(s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^u \bar{Z}_s dW_s, \quad 0 \leq t \leq u, \quad (8)$$

where for $y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$, $z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$, we have

$$\bar{g}(t, y, z) = \begin{pmatrix} g^1(t, y^1 + y^2, z^1 + z^2) - g^2(t, y^2, z^2) \\ g^2(t, y^2, z^2) \end{pmatrix}.$$

The statement (iii) means that the BSDE (8) satisfies BSVP in $K := \{x \in \mathbb{R}^n \mid x \succsim^q 0\} \times \mathbb{R}^n$. Obviously, K is a nonempty, convex closed subset of \mathbb{R}^{2n} . Similarly to the proof of Theorem 2.3, we have

$$\Pi_K(y) = \begin{pmatrix} y^1 + \langle y^1, q \rangle^- q \\ y^2 \end{pmatrix}, \quad d_K^2(y) = (\langle y^1, q \rangle^-)^2, \quad D^2 d_K^2(y) = \begin{pmatrix} 2qq^T I_{\langle y^1, q \rangle < 0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ is the n -order zero matrix. Applying Proposition 2.2, we obtain that the statement (iii) is equivalent to:

$$\begin{aligned} & -4\langle y^1, q \rangle^- \langle q, g^1(t, y^1 + \langle y^1, q \rangle^- q + y^2, z^1 + z^2) - g^2(t, y^2, z^2) \rangle \\ & \leq 2I_{\langle y^1, q \rangle < 0} \sum_{i,j=1}^n q_i q_j (z_i^1)^T z_j^1 + C(\langle y^1, q \rangle^-)^2, \quad P\text{-a.s.}, \end{aligned} \quad (9)$$

where $C > 0$ is a constant independent of (t, y, z) . Let

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} z - z' \\ z' \end{pmatrix},$$

it is then clear that the above inequality (9) becomes (7). The proof of Theorem 3.3 is completed. \square

Furthermore, we can get the following Theorem 3.4.

Theorem 3.4 *If (i) holds, then for any $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$, we have $g^1(t, y, z) \succsim^q g^2(t, y, z)$, P -a.s..*

Proof. Because $q = (q_1, q_2, \dots, q_n)^T \neq 0$, there exists $q_i \neq 0$ for some i . Without loss of generality, we suppose $q_i > 0$. Set $y^k = -\frac{1}{k}e^i$, where the components of e^i are 0 except the i th component which is 1, and k is an arbitrary number in \mathbb{N}^* . Then,

$$\langle y^k, q \rangle^- = \frac{1}{k}q_i, \quad y^k + \langle y^k, q \rangle^- q = \frac{1}{k}[q_i q - e^i].$$

So for any $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$, substituting (y^k, z) and (y, z) into (y, z) and (y', z') in inequality (7) respectively, we have

$$\left\langle q, g^1(t, -\frac{2}{k}e^i + \frac{1}{k}q_i q + y, z) - g^2(t, y, z) \right\rangle \geq -\frac{Cq_i}{4k}, \quad P\text{-a.s.},$$

where $C > 0$ is a constant which does not depend on (t, y, z) . As $k \rightarrow \infty$, it follows from (A2) that $g^1(t, y, z) \succsim^q g^2(t, y, z)$, P -a.s.. The proof of Theorem 3.4 is completed. \square

Note that the converse of Theorem 3.4 is not true (see Example 3.8 given latter). As an application of Theorem 3.4 we immediately have the following corollary.

Corollary 3.5 *If (i) holds for both $q = e^i$, and $q = -e^i$, then for any $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$, we have $g_i^1(t, y, z) = g_i^2(t, y, z)$, P -a.s., where g_i^1 and g_i^2 mean the i th component of g^1 and g^2 respectively.*

Consider some special cases of total order \succsim^q , using Theorem 3.3 or Theorem 3.4, we have the following remarks hold.

Remark 3.6 *Let $n = 1, q = 1$. For this case, $x \succsim^1 y$ means $x \geq y$. Then, (i) is equivalent to $g^1(t, y, z) \geq g^2(t, y, z)$, P -a.s..*

This coincides with 1-dimensional comparison theorem established in [4].

Proof. “ \Rightarrow ”: Immediately from Theorem 3.4.

“ \Leftarrow ”: We only need to prove that (ii) holds. When $y \geq 0$, (ii) obviously holds. So we only consider the case $y < 0$. The left side of inequality (7) equals

$$\begin{aligned} 4y[g^1(t, y', z) - g^2(t, y', z')] &\leq 4y[g^2(t, y', z) - g^2(t, y', z')] \\ &\leq \frac{2}{\mu^2}|g^2(t, y', z) - g^2(t, y', z')|^2 + 2\mu^2 y^2 \\ &\leq 2|z - z'|^2 + 2\mu^2 y^2, \quad P\text{-a.s.} \end{aligned}$$

Letting $C = 2\mu^2$ implies that (ii) holds. The proof of Remark 3.6 is completed. \square

Remark 3.7 *Let $q = e^i$. For this case, $y^1 \succsim^q y^2$ means $y_i^1 \geq y_i^2$, where y_i^1 and y_i^2 represent the i th component of y^1 and y^2 respectively. Then the following statements are equivalent:*

- (iv) *For any $u \in [0, T]$, $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_u, P; \mathbb{R}^n)$ such that $\xi_i^1 \geq \xi_i^2$, then the unique solutions (Y^1, Z^1) and (Y^2, Z^2) in $\mathcal{S}_u^2 \times \mathcal{H}_u^2$ to BSDEs (6) over time interval $[0, u]$ satisfy $(Y_t^1)_i \geq (Y_t^2)_i$, P -a.s., for $t \in [0, u]$;*
- (v) *For any $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$, we have*

$$-4y_i^- [g_i^1(t, y + y_i^- e^i + y', z) - g_i^2(t, y', z')] \leq 2I_{y_i < 0} |z_i - z'_i|^2 + C(y_i^-)^2, \quad P\text{-a.s.},$$

where $C > 0$ is a constant independent of (t, y, z) .

The proof is straightforward by taking $q = e^i$ in Theorem 3.3. From Remark 3.7, consider the comparison theorem for the i th component in multidimensional BSDEs, we can only set $q = e^i$ in Theorem 3.3. It does not need to consider other components. However, if we use the comparison theorem of multidimensional BSDEs in [7] to consider the comparison theorem of the i th component, we have to do some restrictions on other components.

Example 3.8 Let $n = 2$. For any $t \in [0, T]$, $y = (y_1, y_2)^T \in \mathbb{R}^2$, $z \in \mathbb{R}^{2 \times d}$, we have $g^1(t, y, z) = (y_1 + y_2, t)^T$, and $g^2(t, y, z) = (y_1 + y_2 - 1, t)^T$. Obviously, $g^1(t, y, z) \succ^{e^1} g^2(t, y, z)$. However, statement (iv) in Remark 3.7 does not hold for $i = 1$. To prove it, we just suppose that (v) holds for $i = 1$. Then for any $y' = (y'_1, y'_2)^T$, $y = (y_1, -3)^T$, $z = z'$, where y_1 being any negative number. We get $C > -\frac{8}{y_1}$, which contradicts the condition that C is a positive constant independent of y . Thus, (v) dose not hold for $i = 1$, so does (iv).

Remark 3.9 Let $g^1 = g^2 = g$, and $q = e^i$. Then, (iv) is equivalent to that for any $t \in [0, T]$, the i th component of g denoted by g_i depends only on y_i and z_i , P -a.s..

Proof. We only need to show (v) is equivalent that for any t , g_i depends only on y_i, z_i .
“ \Rightarrow ”: In (v), choose $y = -\frac{1}{k}e^i$, where k is an arbitrary number in \mathbb{N}^* . Then for any $y' \in \mathbb{R}^n$, and any $z, z' \in \mathbb{R}^{n \times d}$ such that $z_i = z'_i$, we have

$$-4\frac{1}{k}[g_i(t, y', z)] - g_i(t, y', z') \leq C\frac{1}{k^2}, \quad P\text{-a.s..}$$

Let $k \rightarrow \infty$, we deduce that $g_i(t, y', z) \geq g_i(t, y', z')$, P -a.s.. Similarly, we obtain $g_i(t, y', z') \geq g_i(t, y', z)$, P -a.s.. Hence, $g_i(t, y', z') = g_i(t, y', z)$, P -a.s.. Therefore, for any (t, y) , g_i depends only on z_i , P -a.s..

For any \bar{y} such that $\bar{y}_i = 0$, let $y = \bar{y} - \epsilon e^i, \epsilon > 0$. Then for any $y', z = z'$ in (v), we get

$$-4\epsilon[g_i(t, \bar{y} + y', z) - g_i(t, y', z)] \leq C\epsilon^2, \quad P\text{-a.s..}$$

Letting $\epsilon \rightarrow 0$, we deduce that $g_i(t, \bar{y} + y', z) \geq g_i(t, y', z)$, P -a.s.. Noticing the property of \bar{y} , we also get $g_i(t, \bar{y} + y', z) \leq g_i(t, y', z)$, P -a.s.. Therefore, $g_i(t, \bar{y} + y', z) = g_i(t, y', z)$, P -a.s., that is, for any (t, z) , g_i depends only on y_i , P -a.s..

From the arguments above it follows that for any t , g_i depends only on y_i and z_i , P -a.s..

“ \Leftarrow ”: It's clearly that (v) is always true for $y_i \geq 0$. Now we consider $y_i < 0$, and have

$$\begin{aligned} 4y_i[g_i(t, y - y_i e^i + y', z) - g_i(t, y', z')] &= 4y_i[g_i(t, y'_i, z_i) - g_i(t, y'_i, z'_i)] \\ &\leq \frac{2}{\mu^2}|g_i(t, y'_i, z_i) - g_i(t, y'_i, z'_i)|^2 + 2\mu^2 y_i^2 \\ &\leq 2|z_i - z'_i|^2 + 2\mu^2 y_i^2, \quad P\text{-a.s..} \end{aligned}$$

Thus, letting $C = 2\mu^2$, (v) immediately holds. The proof of Remark 3.9 is completed. \square

Remark 3.10 If (iv) holds for all $i \in \{1, \dots, n\}$, we have

- (vi) for any $u \in [0, T]$, $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_u, P; \mathbb{R}^n)$ such that $\xi^1 \geq \xi^2$, then the unique solutions (Y^1, Z^1) and (Y^2, Z^2) in $\mathcal{S}_u^2 \times \mathcal{H}_u^2$ to BSDEs (6) over time interval $[0, u]$ satisfy

$$Y_t^1 \geq Y_t^2, \quad P\text{-a.s.,} \quad \forall t \in [0, u].$$

The proof of Remark 3.10 is obviously. However, the converse of Remark 3.10 is not true, that is (iv) can not be deduced from (vi). We use the following example to illustrate this.

Example 3.11 Let $n = 2$. Then for any $t \in [0, T]$, $z \in \mathbb{R}^{2 \times d}$, $y = (y_1, y_2)^T \in \mathbb{R}^2$, $g^1(t, y, z) = g^2(t, y, z) = (y_1 + y_2, |z_2|)^T$, due to Theorem 2.2 in [7], (vi) holds. However, from Remark 3.9, (iv) does not hold for $i = 1$.

Remark 3.12 Let $q = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^T = \frac{1}{\sqrt{n}} \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^n$. For this case, it is seen that the statement (i) holds for q is equivalent to the following:

$$\begin{aligned} & -4 \left(\sum_{j=1}^n y_j \right)^- \left[\sum_{i=1}^n \left(g_i^1 \left(y + \frac{1}{n} \left(\sum_{j=1}^n y_j \right)^- + y', z \right) - g_i^2(y', z') \right) \right] \\ & \leq 2I_{\sum_{j=1}^n y_j < 0} \sum_{i,j=1}^n (z_i - z'_i)^T (z_i - z'_j) + C \left[\left(\sum_{j=1}^n y_j \right)^- \right]^2, \quad P\text{-a.s.} \end{aligned}$$

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References

- [1] Aubin, J.P., Da Prato, G. Stochastic viability and invariance. *Ann. Sci. Norm. Pisa.*, 17: 595-613 (1990)
- [2] Buckdahn, R., Quincampoix, M., Rascanu, A. Viability property for a backward stochastic differential equation and applications to partial differential equations. *Probab. Theory Rel.*, 116: 485-504 (2000)
- [3] Chen, Z., Epstein L. Ambiguity, risk, and asset returns in continuous time. *Econometrica*, 70(4): 1403-1443 (2002)
- [4] Coquet, F., Hu, Y., Memin, J., Peng, S. A general converse comparison theorem for backward stochastic differential equations. *C. R. Acad. Sci. Paris, Sér. I Math.*, 333: 577-581 (2001)
- [5] El Karoui, N., Peng, S., Quenez, C. Backward stochastic differential equations in finance. *Math. Finance*, 7(1): 1-71 (1997)
- [6] Hu, Y., Imkeller, P., Müller, M. Utility maximization in incomplete markets. *Ann. Appl. Probab.*, 15(3): 1691-1712 (2005)
- [7] Hu, Y., Peng, S. On the comparison theorem for multidimensional BSDEs. *CR. Math.*, 343: 135-140 (2006)
- [8] Morlais, M.A. Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. *Finance Stoch.*, 13: 121-150 (2009)
- [9] Pardoux, E., Peng, S. Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.*, 14: 55-61 (1990)
- [10] Peng, S. Stochastic Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 30(2): 284-304 (1992)