An Introduction to Gravitational Lensing in TeVeS

HongSheng Zhao^{1,2},

 1 SUPA, University of St. Andrews, KY16 9SS, UK $^2hz4@st-andrews.ac.uk$

Based on presentations at Sicily Lensing School, Oct 29-Nov 3. COMMENTS WELCOME

ABSTRACT

Bekenstein's (2004) TeVeS theory has added an interesting twist to the search for dark matter and dark energy, modifying the landscape of gravity-related astronomy day by day. Built bottom-up rather than top-down as most gravity theories, TeVeS-like theories are healthily rooted on empirical facts, hence immediately passing sanity checks on galaxy rotation curves, solar system constraints, even bullet cluster of galaxies and cosmology with the help of 2eV neutrinos. Nonetheless, empirical checks are far from perfect and complete, and groups of different expertises are rapidly increasing the number of falsifiable properties of the theory. The theory has also been made much simpler and more general thanks to the work of Zlosnik, Ferreira, Starkman (astro-ph/0606039, 0607411). Here I attempt a tutorial of how to compute lensing convergence, time delays etc in TeVeS-like theories for non-spherical lenses. I gave examples to illustrate a few common caveats of Dark-Matter-guided intuitions.

Key words: gravitational lensing—cosmology, gravity

1 INTRODUCTION

1.1 GL in any co-variant metric theory

While gravitational lensing (GL) is a cornerstone of Einsteinian gravity, light propagation is actually well-defined in any generic metric theories of gravity and one can test such theories using lensing data. In fact, light bending is a general property of E&M waves propagating following Fermat's principle, which happens in a non-uniform medium where the effective speed of light c varies (e.g., as in atomospherical seeing) in a flat space-time. Lensing also happens as light with constant speed c following geodicics in vaccum in a curved space-time (e.g., bending by the Sun), but the link between the matter density and the curvature need not be given Einstein's equation.

For example, the gravitational pre-factor $G_{\rm eff}$ needs not be a true constant in galaxies where the gravity is so weak that we lack precise experiments to measure $\frac{G_{\rm eff}m_1m_2}{r^2}$ force. The solar gravity on Pluto 4×10^4 greater than a typical place in a galaxy. E.g., the Sun's acceleration around the Galaxy

$$g \sim \frac{(200 \,\mathrm{km s^{-1}})^2}{10 \,\mathrm{kpc}} \sim \frac{0.1c}{HubbleTime} \sim \frac{1\mathrm{m}}{\mathrm{dav}^2} \sim \frac{1\mathrm{Angstrom}}{\mathrm{sec}^2}.(1)$$

A mundane example of 1Angstrom per second squared gravity is the mutual Newtonian gravity of two nearly parallel sheets of printing papers approximately. It is well-known that a parallel plate electric capacitor yields a different Efield if immersed in vaccum entirely (Casmir effect), or filled in a dielectric air inside. The gravitational attraction of two sheets of paper could depend on environment in a similar way for very different physics. Consider a *Gedanken* experiment with a gravitationally torquing pendulum made by two misaligned suspended sheets of paper. If one could measure the period of the torquing pendumlum not only here on Earth (as in free-fall experiments in an Einstein tower), but also take the table-top experiments to the edge of the solar system (where Pioneer 10/11 probes are), in the interstellar space (where galactic stars orbit) and in the expanding void between galaxies, then one could measure how G_{eff} changes with space and time.

1.2 The three pillars of the standard cosmology

The standard cosmological paradigm is built on three pillars: Cold Dark Matter, a cosmological constant, and Einsteinian gravity. While independent experimental basis of each of the three is debatable on astronomical scales, but their synergy (characterised by the cosmological pie) has proven amazingly successful at describing the Universe especially on large scale.

Despite its apparantly enticing simplicity, the paradigm has much to be understood and is under pressure to be modified by observations of galaxy scale and by competing theories like TeVeS. For example, the experimentally undetected cold dark matter (generally thought to be Super-Symmetry particles) is predicted to clump in scale-free fashion, while observations of dwarf galaxies suggest a kpc-scale free-streaming length of dark matter particles. The idea of introducing a constant or a vaccum energy 100 orders of magnitude lower than what the SUSY physics can provide naturally is still regarded by many theoreticians as unsatisfactory.

These fine-tunings have lead some to believe the paradigm is an effective theory, e.g., a 4D projection of a more fundamental 5D brane world theory. Some also question the Einsteinian gravity since its associated equivalence principles, remain untested on galaxy scale and cosmological scale.

1.3 Modified gravity: motivations and history

Modifying gravity is a reoccuring excercise which started ever since the general acceptance of Einsteinian gravity, which was itself a revolutionary modification to Newtonian gravity. Many theories modify the Einstein-Hilbert action to introduce a new scalar field which manifests itself only through the extra bending of space time, but its coupling to the metric is different from the simple coupling of massive particles with the space-time metric.

By construction the theories would respect Special Relativity prescription of metric co-variance, and preserve conservations of momentum and energy. They do allow for a table-top Cavendish-type experiment with a torquing pendulum to measure an effective gravitational constant $G_{eff}(t,x)$ which varies with time and environment of the experiment. For example, the recent motivation to replace the cosmological constant in General Relativity leads to theories with G_{eff} depending on the curvature of space-time, which evolves with the cosmic time in a way to drive the acceleration of the universe at late time.

However, among two dozen theories proposed after GR, very few survive the precise tests on SEP in the solar system and the well-studied binary pulars. Even fewer are motivated and succeeded in addressing both astronomical dark matter and cosmological constant.

1.4 The secret of TeVeS and variants

TeVeS is an exception. It holds the promise of explaining both dark matter and cosmological constant by relaxing the SEP (strong equivalence principle) only in untested weak gravity envionments like in galaxies, but respecting the SEP to high accuracy in the solar system.

Crudely speaking such theory has an aether-like field with an aquadratic kinetic term in its Lagrangian density, so the G_{eff} can be made a function of the strength of gravity |g|, such that G_{eff} is constant within 10^{-16} anywhere in the solar system, yet varies by a factor of 10 in galaxies and in the universe over a Hubble time where |g| is much smaller. Enhancing the G_{eff} mimics the effects of adding dark matter, and reducing the G_{eff} can drive the acceleration of the universe.

2 A CHARACTERISTIC SCALE FOR DARK MATTER

As one of the important issue to be understood about dark matter, it has long been noted that on galaxy scales dark matter and baryonic matter (stars plus gas) have a remarkable correlation, and respects a mysterious acceleration scale a_0 (Milgrom 1983, McGaugh 2005).

The Newtonian gravity of the known matter (baryons etc.) \mathbf{g}_k and the dark matter gravity \mathbf{g}_{DM} are correlated through an empirical relation (Zhao and Famaey 2006, Angus, Famaey, Zhao 2006, Famaey, Gianfranco, Bruneton, Zhao 2006) such that the light-to-dark ratio, experimentally determined to fit rotation curves, satisfies

$$\frac{g_k}{g_{DM}} = \frac{g_{DM} + \alpha g_k}{a_0},\tag{2}$$

where $0 \leq \alpha \leq 1$ is a parameter. Let $\alpha = 0$ we get a very simple relation

$$g_{DM} \approx \sqrt{g_k a_0}, \qquad a_0 \equiv 1 Angstrom sec^{-2}$$
 (3)

where a_0 is the forementioned gravity scale, below which DM and DE phenomena start to surface.

Such a tight correlation is difficult to understand in a galaxy formation theory where dark matter and baryons interactions enjoy a huge degrees of freedom. This spiral galaxy based empirical relation is also consistent with some elliptical galaxies and gravitational lenses.

3 A SCALE FOR DARK ENERGY

Equally peculiar is the amplitude of vaccum energy density Λ , which is of order 10^{120} times smaller than its natural scale. It is hard to explain from fundamental physics why vaccum energy starts to dominate the Universe density only at the present epoch, hence marking the present as the turning point for the universe from de-acceleration to acceleration.

This is related to the fact that

$$a_0 \sim \sqrt{\Lambda} \sim cH_0$$
 (4)

where a_0 is the characteristic scale of DM as well.

Somehow dark energy and dark matter are tuned to shift dominance when the energy density falls below $\frac{a_0^2}{8\pi G}$. These empirical facts should not be completely treated as random coincidences of the fundamental parameters of the universe. The explanation with standard paradigm has been unsatisfactory.

4 THE METRIC AND DYNAMICS OF THE TEVES FIELDS

TeVeS, as GR, is a metric theory. Let $g_{\mu\nu}$ being the physical coordinates, then near a quasi-static system like a galaxy, the physical space-time is only slightly curved, and can be written as in a rectangular coordinate $\boldsymbol{x} = (x_1, x_2, x_3)$ centred on the galaxy as

$$-c^2 d\tau^2 = g_{tt} dt^2 + g_{rr} dl^2, \qquad (5)$$

$$dl^2 = (dx_1^2 + dx_2^2 + dx_3^2). (6)$$

Introducing a small quantity $\frac{|\Phi|}{c^2} \ll 1$, we can write the metric components

$$g_{rr} \approx -c^2 g_{tt}^{-1} \approx 1 - \frac{2\Phi}{c^2}.$$
(7)

To show that $\Phi(\mathbf{x})$ takes the meaning of a gravitational potential, we note that a non-relativistic massive particle moving in this metric follows the geodesic

$$\frac{d^2 x_i}{dt^2} - \frac{\partial g_{tt}}{2\partial x_i} \approx 0, \qquad \rightarrow \qquad \frac{d^2 x}{dt^2} \approx -\nabla \Phi(x), \tag{8}$$

which is the equation of motion in the non-relativistic limit where $\frac{dt}{d\tau} \approx 1$.

4.1 Vector or scalar

Near a quasi-static system like a galaxy, $g_{00} = -(1 + 2\Phi)$, where we omit the factor c^2 for clarity. TeVeS predicts a time-like vector field with four components, which are approximated as

$$A^{\alpha} = (1 - \phi - \Phi, 0, 00) \tag{9}$$

and

 $A_{\alpha} = -(1 - \phi + \Phi, 0, 00) \tag{10}$

to the lowest order, where ϕ is a scalar field.

For most of the system that we are interested, the key is the module of A, which is equivalently described by the scalar field ϕ related through

$$A^2 \equiv g_{ab}A^a A^b \equiv -e^{-2\phi} < 0 \tag{11}$$

This shows the vector field is more fundamental than the scalar field in TeVeS and TeVeS is can be described the physical metric and vector field alone. The original proposal of Bekenstein contains two metric, while most recent work of Zlonik, Ferreira, Starkman (2006, PRD. 74, 0404037) shows that the theory is equally described by a single physical metric $g_{\mu\nu}$, whose geodicics particles and light will follow. The other metric (called Einstein metric) is fully described once the vector field is specified.

4.2 TeVeS as dark matter

In TeVeS, the galaxy potential Φ comes from two parts,

$$\Phi = \Phi_{kn} + \phi \tag{12}$$

where the known Newtonian gravitational potential $\Phi_k(\mathbf{x})$ of known matter of density $\rho_k(\mathbf{x})$ satisfies

$$\nabla \cdot \nabla \Phi_{kn} = 4\pi G \rho_{kn} \tag{13}$$

and the added scalar field satisfies

$$\nabla \left[\mu_s \nabla \phi\right] = 4\pi G \rho_{kn},\tag{14}$$

where

$$\mu_s = \left|\frac{\nabla\phi}{a_0}\right| + O\left(\left|\frac{\nabla\phi}{a_0}\right|^2\right). \tag{15}$$

The picture to keep in mind is that the scalar field replaces the usual role of the potential of the Dark Matter. The vector field A is fully specified once ϕ and Φ are given.

To illustrate how the scalar equation come from Lagrangian of the vector field theory, let's consider a toy model.

5 AN E&M-LIKE 4-VECTOR POTENTIAL

Zlonik, Ferreira, Starkman (2006, astro-ph/0607411) generalised TeVeS as part of a broader class of Einstein-Aether theories, which we will follow here. We will neglect a certain Lagrangian multipler for normalisation of vector field and also make simplifications where possible (setting $c_1 = c_3 =$ -1 and $c_2 = 0$ in their notation, and choosing the simplest modification to the Langrangian). We emphasize the similarity of the 4-vector field here with the 4-vector potential (A_0, A_1, A_2, A_3) in electromagnetism.

Let the vector field $A_{\alpha} = g_{\alpha\beta}A^{\beta}$ be a time-like unit vector with the constrain equation

$$A_{\alpha}A^{\alpha} = -1 \tag{16}$$

and let it be coupled to the metric $g_{\alpha\beta}$, so that the system is governed by an action S or Lagrangian density L,

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + L_{kn} + \left(1 - \frac{2f}{3} + O(f^2) \right) I_{k} \right]^2$$

where R the usual Ricci scalar of the metric \mathbf{g} , L_{kn} is the known matter Lagrangian density coupling the known matter with metric but not coupled to the vector field directly. The 3rd term is the local Lagrangian density of the vector field, which apart from a dielectric-like modification factor $(1 - 2f/3 + O(f^2))$, is the normal electromagnetism-like kinetic coupling to metric

$$L_f = \frac{a_0^2 f^2}{32\pi G} \equiv \frac{F_{\alpha\beta} F^{\alpha\beta}}{32\pi G} \tag{18}$$

where the dimensionless f parameter is a measure of the strength of the Maxwell tensor field $F_{\alpha\beta}$:

$$F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}.$$
(19)

which is the covariant derivative of the 4-potential A_{α} , (as for the electric and magnetic field in electromagnetism).

5.1 Einstein eq. and vector field eq. of motion

Taking variations of the action respect to the metric $g_{\alpha\beta}$ and A_{α} respectively we get the gravitational field equations and vector equation of motion for this theory respectively. Combining the two we have

$$\frac{G_{\alpha\beta}}{8\pi G} = T^{kn}_{\alpha\beta} + T_{\alpha\beta}, \qquad (20)$$

where the left-hand side is the proportional to the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$ and on rhs the 1st term is the stress-energy tensor of known matter, the 1st term is the stress-energy tensor for the vector field $T_{\alpha\beta}$,

$$T_{\alpha\beta} = \tilde{T}_{\alpha\beta} + \frac{1}{8\pi G} A_{\alpha} \nabla_{\mu} [(f + O(f^2))g^{\mu\nu}F_{\nu\beta}], \qquad (22)$$

which is a non-linear function of derivatives of the field A_{β} . Note how the vector field creates the effect of additional matter.

Near a galaxy, $G_{00} = 2\nabla \nabla \Phi$, so the 00th moment of the above equation reduces to

$$\nabla(\mu_s \nabla \Phi) = 4\pi G \rho_{kn},\tag{23}$$

where

4 H. Zhao

$$\mu_s = f + O(f^2), \qquad f = \left|\frac{\nabla\Phi}{a_0}\right|. \tag{24}$$

This way we recover the classical MOND equation of Bekenstein and Milgrom (1984) in the weak field limit $f \to 0$, i.e., the gravity $\nabla \Phi$ drops as $\sqrt{GMa_0}/r$ far away from a point mass M.

5.2 Hubble expansion equation

For homogeneous flat cosmology, we can set the metric

$$ds^{2} = -c^{2}dt^{2} + a(t)^{2}(dr^{2} + r^{2}d\theta^{2} + r^{2}\sin\theta^{2}d\phi^{2})$$
(25)

and the vector field

 $A^{\alpha} = -A_{\alpha} = (1, 0, 0, 0), \tag{26}$

and its derivatives

$$f = 0. \tag{27}$$

The Einstein equation reduces to the following eq. for Hubble expansion

$$\left(\frac{da}{adt}\right)^2 = \frac{8\pi G}{3}\rho_{kn}.$$
(28)

Interestingly this is not modified from the GR, where ρ is the density of known matter of the uniform background.

The actual theory of TeVeS and Zlonik et al. is more sophisticated than above toy model. E.g., Zhao & Famaey (2006) proposed to modify TeVeS with a $\mu = f/(1+f)$, hence the Lagrangian $\int (1-\mu) d \frac{a_0^2 f^2}{32\pi G} = (f-\ln(1+f)) \frac{a_0^2}{16\pi G} =$ $(1-\frac{2f}{3}+O(f^2))L_f$, such that we recover Einstein-Newtonian in strong gravity, and achieve a good fit to rotation curves in the intermediate regime, and have a smooth transition between galaxies and cosmology. The term f^2 or L_f can contain derivatives other than the Maxwell tensors. It is also possible to modify the homogeneous cosmology in TeVeSlike theory to create the effect of dark energy. The cosmology prescribes distance-redshift relations, essential for lensing. For these we assume the background cosmology in TeVeSlike theories must fit the same SNe distance data, hence the resulting distance-redshift relation (at least at redshifts of order unity) is exactly as in LCDM. A detailed study to this effect is shown in Zhao (2006, astro-ph/0610925).

6 THE INTERPOLATING FUNCTIONS IN MOND VS TEVES

In the classical MOND a spherical galaxy of rotation curves V(r) could be explained by modifying gravity:

$$g = \frac{V^2}{r} = \frac{GM}{r^2 \mu(g/a_0)},$$
(29)

where $\mu(g/a_0)$ is the effective "dielectric constant", depending on the gravitational field strength, g. The standard interpolating function

$$\mu(x) = \frac{x}{\sqrt{1+x^2}} \tag{30}$$

is often used in fitting rotation curves. But Zhao & Famaey (2006) argued that this function has undesirable features in TeVeS. Instead they (Angus, Famaey, Zhao 2006) proposed to use

$$\mu(x) = \frac{2x}{1 + (2 - \alpha x) + \sqrt{(1 - \alpha x)^2 + 4x}}$$
(31)

which is a parametric α -family, which recovers Bekenstein's (2004) toy model and the simple model of Famaey & Binney (2005) by setting $\alpha = 0$ and $\alpha = 1$.

The gravitational potential in MOND theory satisfies a modified Poisson's equation,

$$\nabla[\mu\nabla\Phi] = \nabla^2\Phi_N = 4\pi G\rho_{kn} \tag{32}$$

where the ρ_{kn} is the density of all known matter. This is different from TeVeS, where the total potential is the sum of Newtonian potential (Φ_N) and a potential due to a scalar field (ϕ_s):

$$\Phi = \Phi_N + \phi_s. \tag{33}$$

We can see that the scalar potential plays the role of the dark matter gravitational potential, and the Poisson-like equations for the scalar field relates it to the Newtonian potential Φ_N (generated by the baryonic matter),

$$\nabla[\mu_s \nabla \phi_s] = \nabla^2 \Phi_N = 4\pi G\rho \tag{34}$$

where μ_s is a function of the scalar field strength $g_s = |\nabla \phi_s|$, and is derived from a free function in the action of the scalar field. In spherical symmetry, we have

$$\mu_s g_s = \mu(g_s + g_N) = g_N \tag{35}$$

where μ is the interpolating function of MOND, hence the two interpolation function is related by

$$\mu_s = \frac{\mu}{1-\mu}.\tag{36}$$

7 LIGHT BENDING IN SLIGHTLY CURVED SPACE TIME

Light rays trace the null geodesics of the space time metric. Lensing, or the trajectories of light rays in general, are uniquely specified once the metric is given. In this sense light bending works *exactly* the same way in any relativistic theory as in GR.

Near a quasi-static system like a galaxy, the physical space-time is only slightly curved. Consider lensing by the galactic potential $\Phi(\mathbf{r})$. A light ray moving with a constant speed c inside follows the null geodesics $dt = \sqrt{-\frac{g_{rr}}{g_{tt}}} dl$. An observed light ray travels a proper distance $l_{os} = l_{ls} + l_{ol}$ from a source to the lens and then to an observer. Hence it arrives after a time interval (seen by an observer at rest with respect to the lens)

$$\int dt = \int_0^{l_{os}} \frac{dl}{c} - \int_0^{l_{os}} \frac{2\Phi(\mathbf{r})}{c^2} \frac{dl}{c}$$
(37)

containing a geometric term and a Shapiro time delay term due to the Φ potential of a galaxy.

In fact, gravitational lensing in TeVeS recovers many familiar results of Einstein gravity in (non-)spherical geometries. Especially an observer at redshift z = 0 sees a delay $\Delta t_{\rm obs}$ in the light arrival time due to a thin deflector at $z = z_l$

$$\frac{c\Delta t_{\rm obs}(\mathbf{R})}{(1+z_l)} \approx \frac{D_s}{2D_l D_{ls}} \left(\mathbf{R} - \mathbf{R}_s\right)^2 - \int_{-\infty}^{\infty} dl \frac{2\Phi(\mathbf{R}, l)}{c^2}, \qquad (38)$$

as in GR for a weak-field thin lens, $\Phi/c^2 \ll 1$. A light ray penetrates the lens with a nearly straight line segment (within the thickness of the lens) with the 2-D coordinate, $\mathbf{R} = D_l \boldsymbol{\theta}$, perpendicular to the sky, where $D_l(z_l) = l_{ol}/(1 + z_l)$ is the angular diameter distance of the lens at redshift z_l , D_s is the angular distances to the source, and D_{ls} is the angular distance from the lens to the source. The usual lens equation can be obtained from the gradient of the arrival time surface with respect to \mathbf{R} . i.e.,

$$x - \frac{D_l D_{ls}}{D_s} \alpha_x(x, y) = x_s$$

$$y - \frac{D_l D_{ls}}{D_s} \alpha_y(x, y) = y_s$$
(39)

where the deflection

$$\alpha_x = \int_{-\infty}^{\infty} dl \frac{2\partial_x \Phi(x, y, l)}{c^2}, \qquad (40)$$

$$\alpha_y = \int_{-\infty}^{\infty} dl \frac{2\partial_y \Phi(x, y, l)}{c^2},$$

and the convergence

$$\kappa = \frac{D_l D_{ls}}{2D_s} \left(\partial_x \alpha_x + \partial_y \alpha_y \right),\tag{41}$$

and likewise for the shear,

$$\gamma_2 = D_l \partial_y \alpha_x,\tag{42}$$

and

$$\gamma_1 = \frac{D_l}{2} \left(\partial_x \alpha_x - \partial_y \alpha_y \right) \tag{43}$$

and for the amplification A

$$A^{-1} = (1 - \kappa)^2 - \gamma_1^2 - \gamma_2^2.$$
(44)

The time delay between a pair of images i and j is given by the path integral

$$\frac{c\Delta t_{\rm obs}^{ij}}{(1+z_l)} = \int_{p_i}^{p_j} dp \, L(p), \tag{45}$$

$$L(p) \equiv \frac{dx}{dp} \left(\frac{\alpha_x(p_i) + \alpha_x(p_j)}{2} - \alpha_x(p) \right)$$

$$+ \frac{dy}{dp} \left(\frac{\alpha_y(p_i) + \alpha_y(p_j)}{2} - \alpha_y(p) \right),$$
(46)

where x(p), y(p) defines a path from image *i* to image *j* as *p* varies from p_i to p_j .

8 SUBTLE DIFFERENCES OF TEVES LENSING AND DM LENSING

8.1 κ of TeVeS and DM

An interesting point is that in GR κ is proprotional to the projected surface density of known matter. This is *not the case for a non-linear theory*. We can express κ into of the critical density as follows,

$$\kappa = \frac{\Sigma(x, y)}{\Sigma_{crit}}, \qquad \Sigma_{crit}^{-1} \equiv \frac{4\pi G D_l D_{ls}}{D_s c^2}, \tag{47}$$

where we define an effective projected density as follows,

note the integrand is NOT the true matter volume density at (x,y,l), rather

$$\tilde{\rho}(x,y,l) \equiv \frac{\nabla^2 \Phi(x,y,l)}{4\pi G} = \rho_{kn} + \rho_{eDM} > \rho_{kn} \tag{49}$$

because Φ is addition of two fields, and we have an effective Dark Matter (eDM) from the ϕ field,

$$\rho_{eDM} = \frac{\nabla^2 \phi(x, y, l)}{4\pi G} \tag{50}$$

The eDM tracks the known matter, because the TeVeS ϕ field is determined by non-linearly with ρ_k .

$$\rho_{kn} = \frac{\nabla f \nabla \phi(x, y, l)}{4\pi G} = \frac{\nabla^2 [\Phi - \phi(x, y, l)]}{4\pi G}$$
(51)

E.g., a TeVeS point lens has a non-zero convergence due to the non-zero effective DM halo. So important differences between lensing in TeVeS and in GR is in the predicted metric or potential Φ for a given galaxy mass distribution ρ_k .

In general, our non-linear Poisson equation can be solved by adapting the code of Ciotti, Nipoti, Pasquale (2006). In some cases, one can also take the potential-todensity, and start with a reasonable guess for the potential, and find the density by taking appropriate derivatives, e.g., the application of Angus et al. (2006a, 2006b) on the bullet cluster. In special cases, one can also solve the TeVeS Poisson equations analytically. This is the case for Kuzmin disks.

8.2 κ in a non-spherical model

Here we illustrate how to model non-spherical lens galaxy in TeVeS, and point out some subtle difference with DM. We present models with identical rotation curves, one in TeVeS and one in DM, demonstrate the subtle difference in their lensing signal.

To keep the calculations tractable, we approximate the lens potential as that of an edge-on Kuzmin disk with a density

$$\rho(R,Z) \equiv \frac{Mb\delta(Z)}{2\pi (R^2 + b^2)^{3/2}}$$
(52)

in cylindrical coordinates around the symmetry axis (R, Z) of the Kuzmin disk of total mass M. We work out the bending angle and the lens equation in the sky directions x and y. Here the Newtonian gravity **g** is given by,

$$-(g_R^N, g_Z^N) = \frac{GM}{\left[R^2 + (Z+b_{\pm})^2\right]^{3/2}} (R, Z+b_{\pm})$$
(53)

where $b_{\pm} = \pm b$ for positive or negative Z respectively, because the upper part of a Kuzmin disk is in a spherical potential like that of a point-mass centered on (x_0, y_0, z_0) below the disk. The potential of the lower part of the Kuzmin disk is a mirror-image of the upper part. The Kuzmin scalelength $b = \sqrt{x_0^2 + y_0^2 + z_0^2}$ is smaller than the half-light radius r_{half} by a factor $\sqrt{3}$.

The nice thing about a Kuzmin disk is that the MONDian gravity is parallel to the Newtonian gravity riigourously because the equal potential contours coincide with equal Newtonian gravity contours. Assuming the Bekenstein mu-function, or

$$\mu_s = \frac{g^s}{a_0}, \qquad \mu_s \mathbf{g}^s = \mathbf{g}^N, \tag{54}$$

we get the TeVeS scalar field

$$-(g_R^s, g_Z^s) = \frac{v_0^2}{R^2 + (Z + b_{\pm})^2} (R, Z + b_{\pm})$$
(55)

where

$$v_0^2 \equiv \sqrt{GMa_0}.\tag{56}$$

One can easily verify that

$$-\nabla \left[\left| \frac{\mathbf{g}^s}{a_0} \right| \mathbf{g}^s \right] = 4\pi G\rho, \tag{57}$$

One can also replace the scalar field with a spherical DM halo potential $v_0^2/2\ln(R^2+Z^2+b^2)$ with a gravity

$$-(g_R^{DM}, g_Z^{DM}) = \frac{v_0^2}{R^2 + Z^2 + b^2}(R, Z)$$
(58)

such that they two descriptions give identical gravity g_R on the equtorial Z = 0 plane where rotation curves are measured. So the circular velocity curve is

$$V_{\rm cir}^2 = Rg_R(R,0) == \frac{GMR^2}{\left[R^2 + b^2\right]^{3/2}} + \frac{v_0^2 R^2}{R^2 + b^2}$$
(59)

which is rising as solid-body at centre and asymptotically flat velocity v_0 at large radii. Far away from the disk plane, TeVeS resembles a spherical DM halo very well (Read & Moore 2005).

Nonetheless there is a subtle difference. While the DM force is centrally-pointing, the scalar force is pointed more to a point below/above the disk. At large radii R, and small |Z|, we have

$$|g_Z^s| - |g_Z^{DM}| \sim \frac{b\sqrt{GMa_0}}{R^2 + b^2}, \ |\mathbf{Z}| \ll \mathbf{b} < \mathbf{R}$$
 (60)

so the TeVeS scalar field provides *stronger vertical force* than the spherical DM close to the disk plane.

8.3 Lensing by Kuzmin Disk

Assuming Bekenstein's μ function, we can get the lens equations,

$$x - x_s = \frac{D_l D_{ls}}{D_s} \alpha_x = (x - x_0)k_1 + (x + x_0)k_2,$$
(61)

$$y - y_s = \frac{D_l D_{ls}}{D_s} \alpha_y = (y - y_0)k_1 + (y + y_0)k_2,$$
(62)

where (x_s, y_s) are the source positions (projected on the lens plane), k_1 and k_2 are dimensionless functions of the baryon mass, distances and inclinations.

Consider the simplest case we have a face-on Kuzmin disk with the parameters $x_0 = y_0 = 0$, $z_0 = b$. For this axisymmetric lens, deflection points to the lens origin and is a function of R only. One can predict convergence and shear by taking appropriate derivatives. The critical (Einstein) ring can be obtained by setting $x_s = y_s = 0$ so that the Jacobian $A^{-1} = \frac{dx_s dy_s}{dx dy} = 0$.

 $R_M \equiv \sqrt{\frac{4GM}{c^2} \frac{D_l D_{ls}}{D_s}}, \qquad D_0 \equiv \frac{c^2}{a_0} \approx \frac{6c}{H_0}$ (63)

be the Einstein ring size in GR and a characteristic MOND distance respectively, then TeVeS predicted critical radius R satisfies

$$1 = \frac{D_l D_{ls}}{D_s} \frac{\alpha(R)}{R} = 2(k_3 + k_4), \tag{64}$$

where

$$2k_3 \equiv \frac{R_M^2}{R^2} \frac{\chi^2}{1 + \sqrt{1 - \chi^2}}$$
(65)

$$2k_4 \equiv (2 \arcsin \chi) \frac{R_M}{R} \frac{D_l D_{ls}}{D_s D_0}$$
(66)

$$2k_5 \equiv (\pi\chi)\frac{R_M}{R}\frac{D_l D_{ls}}{D_s D_0} \tag{67}$$

$$\chi \equiv \frac{R}{\sqrt{R^2 + b^2}},\tag{68}$$

where we have used $k_1 = k_2 = k_3 + k_4$.

The corresponding face-on Kuzmin disk plus spherical DM model predicts

$$1 = \frac{D_l D_{ls}}{D_s} \frac{\alpha(R)}{R} = 2(k_3 + k_5), \tag{69}$$

Hence MOND can create an illusion of an isothermal halo lens with a terminal velocity $v_0^2 = \sqrt{GMa_0}$ through the terms with k_4 and k_5 .

Note $k_5 \geq k_4$. This means we have a slightly *bigger* critical radius for the case with DM halo vs. the case with TeVeS. Likewise TeVeS would predict slightly different time delay between images than DM halos for a fixed H_0 . These are somewhat surprising since the two models have *identical rotation curves*. This suggests that a combination of lensing and kinematics will be able to differentiate DM and TeVeS.

8.4 MONDian effects on critical rings

To understand the correction due to MOND, let's consider lensing by a spherical point lens, which can be obtained by letting b = 0, hence $\chi = 1$,

$$2k_3 = \frac{R_M^2}{R^2}, \qquad 2k_4 = 2k_5 = \frac{\pi R_M}{R} \frac{D_l D_{ls}}{D_0 D_s}$$
(70)

It is helpful to estimate rescaled lens-source distance in TeVeS

$$\frac{D_l D_{ls}}{D_0 D_s} \sim \frac{D_l D_{ls}}{25 \text{Gpc} D_s} \ll 0.1 \tag{71}$$

for all lenses and sources. Hence $2k_4 \leq 2k_5 \sim \frac{D_l D_{ls}}{D_0 D_s} \ll 2k_3 \sim 1$ near the critical ring $R \sim R_M$, so the MON-Dian effect is never very important near critical line. This means critical rings (or strong lensing) can only occur in line of sight which pass through regimes of strong gravity $R \sim R_M \ll \sqrt{\frac{GM}{a_0}}$ where *the MONDian effect is small*.

9 LENSING TESTS OF TEVES

For lenses with almost co-linear double images in the CAS-TLES survey, Zhao, Bacon, Taylor, Horne (2006) conducted a detailed fit using spherical point or Hernquist profile

Let

lenses. Cares have been taken in including the K-correction, the luminosity evolution with redshift, and the possibility of significant gas and extinction from dust. They applied two methods, using the image positions only, and using the image amplifications. They found that the mass-to- M_* ratios calculated using the two independent methods closely agree, and all but two of the lenses are found to have M/M_* between 0.5 and 2. This shows that TeVeS is a sensible theory for doing gravitational lensing.

However, the authors caution that there are several lenses in galaxy clusters which require extremely high M/L. Clearly more detailed models are needed, including flattening and the cluster environment. As a first attempt in this direction, Angus et al. (2006) found that the lensing peaks of the Bullet Cluster could be explained by adding neutrinos in a TeVeS-like modified gravity; the phase space density of neutrinos at the lensing peaks requires 2eV mass to in order not to violate exclusion principle for fermions. In general gravitational lensing can be used as a useful approach to distinguish between theories of gravity, and to probe the functional form of the modification function μ .

10 OTHER SANITY CHECKS OF TEVES

Sanity checks from small to large scale have also been done in recent papers. TeVeS is found to be

• OK with solar system (Bekenstein & Maguijo 2006)

• OK with Milky Way and Bulge Microlensing (no cusp problem, Famaey & Binney 2006)

• Excellent description of spiral rotation curves (Mc-Gaugh 2005, Famaey et al. 2006)

• OK with elliptical galaxies lenses (Zhao, Bacon, Taylor, Horne 2006)

• OK with galaxy clusters if with neutrinos (Angus, Shan, Zhao, Famaey, 2006),

• TeVeS universe can accelerate (Zhao 2006, astro-ph/0610056)

• Structures and CMB can form from linear perturbations (Dodelson & Liguori 2006).

TeVeS is by no means a firmed established paradigm since many comparisons of the theory with observations are still unknown, but in the process of understanding and falsifying TeVeS, we learn to design clever dark matter models and appreciate better the robustness of GR. As it stands,

• TeVeS is not grossly inconsistent with observations of lensing apart from a few outliers associated with galaxy clusters where massive neutrinos would contribute to the deflection of the light,

• CMB anisotropy are predictable (Skordis et al. 2005),

• structure formation in non-linear potential can in principle be followed by N-body codes (Ciotti et al. 2006).

ACKNOWLEDGEMENTS

I thank Benoit Famaey, Pedro Ferriera, Tom Zlonik, Constantinous Skordis, David Mota, Luca Ciotti, Carlo Nipoti Huanyuan Shan, Daming Chen for discussions on cosmology, dynamics and lensing. The material of this presentation draws from following papers:

REFERENCES

- Angus G., Shan H, Zhao H, Famaey B. 2006, ApJ Letter, in press (astro-ph/0609125)
- Bekenstein J. 2004, Phys. Review D., 70, 3509
- Bekenstein J. & Maguijo J. 2006, Phys. Review D.,
- Bradac M. et al. 2006, ApJ, in press (astro-ph/0608408)
- Ciotti L., Londrillo P., Nipoti C. 2006, ApJ, 640, 741
- Clowe D., et al. 2006, ApJ, 648, L109
- Famaey B., Binney J. 2005, 363, 603
- Famaey B., Gianfranco G., Bruneton J.P., Zhao H. 2006, Phy.Rev.D, (astro-ph/0611227)
- Milgrom M. 1983, ApJ, 270, 365
- McGaugh S. 2005, PRL
- Read J, Moore B. 2005, MNRAS, 361, 971
- Zhao H., Bacon D., Taylor A., Horne K. 2006, MNRAS, 368, 171
- Zhao H., review for "Quantum to cosmology: fundamental Physics in Space", Washinton, astro-ph/0610056
- Zhao H. Famaey B. 2006, ApJ Letter, 638, L9
- Zlonik, Ferreira, Starkman 2006, Physical Review Letter, (astroph/0607441).
- Zlonik, Ferreira, Starkman 2006, Physical Review D., 74, 0404037